

CONTACT CUTS

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ABSTRACT. We describe a contact analog of the symplectic cut construction [L]. As an application we show that the group of contactomorphisms for certain overtwisted contact structures on lens spaces contains countably many non-conjugate two tori.

1. INTRODUCTION

In this paper we introduce the notion of cuts in the contact category and use it to study the group of contactomorphisms of certain overtwisted contact structures on 3-manifolds.

Recall that a contact form on a $2n + 1$ dimensional manifold M is a 1-form α such that

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Thus a contact form defines an orientation on M . Note that if n is odd then α and $-\alpha$ define the same orientation. It is not hard to see that if f is a nowhere vanishing function on the manifold M and α is a contact form then $f\alpha$ is also a contact form. The kernels of α and $f\alpha$ are, of course, the same. A **co-oriented contact structure** on a contact manifold M is a subbundle ξ of the tangent bundle TM which is given as the kernel of a contact 1-form. A **contact structure** ξ on M is a subbundle of TM which is locally, but not necessarily globally, the kernel of a contact form.

Throughout the paper α will always denote a contact form and ξ will always denote a contact structure. Whenever convenient we will refer to a pair (M, α) or to a pair (M, ξ) as a contact manifold.

A diffeomorphism of a contact manifold (M, ξ) preserving the contact structure ξ is called a **contactomorphism**.

A contact 3-manifold (M, ξ) is **symplectically fillable** if there exists a compact symplectic manifold (W, ω) with boundary such that

1. $\partial W = M$,
2. $\omega|_{\xi}$ does not vanish, and
3. the orientation of M defined by ξ agrees with the orientation of M as the boundary of the symplectically oriented manifold W .

One often says in this case that ξ is fillable.

A contact structure ξ on a 3-manifold M is **overtwisted** if there is an embedded 2-disk $D \subset M$ such that the boundary ∂D is tangent to ξ and D is transverse to ξ along ∂D . A contact structure on a 3-manifold which is not overtwisted is called **tight**. A theorem due to Gromov and Eliashberg shows that a fillable contact structure is tight [E4, Theorem 3.2] (see also [Gr, E2]). Thus the standard contact structure on S^3 induced by its embedding in \mathbb{C}^2 as the unit sphere (the contact distribution consists of complex lines tangent to the sphere) is tight.

Eliashberg showed [E1] that on closed 3-manifolds overtwisted contact structures are classified by the homotopy classes of the corresponding plane fields. We will exploit this property of overtwisted contact structures to show that for a class of such structures the corresponding groups of contactomorphisms contain countably many non-conjugate 2-tori.

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2. CUTS

2.1. Topological cuts. We start with the following observation.

Proposition 2.1. *Consider a smooth action $(\lambda, m) \mapsto \lambda \cdot m$ of a circle S^1 on a smooth manifold M . Let $f : M \rightarrow \mathbb{R}$ be an S^1 invariant function. Suppose that $a \in \mathbb{R}$ is a regular value of f and suppose that the action of S^1 on $f^{-1}(a)$ is free. Then the topological space*

$$M_{[a, \infty)} := \{m \in M \mid f(m) \in [a, \infty)\} / \sim,$$

where, for $m \neq m'$, $m \sim m'$ if and only if

1. $f(m) = f(m') = a$ and
2. $m = \lambda \cdot m'$ for some $\lambda \in S^1$,

is a smooth manifold. The set $\{m \in M \mid f(m) > a\}$ is open and dense in $M_{[a, \infty)}$ and the difference $M_{[a, \infty)} \setminus \{m \in M \mid f(m) > a\}$ is diffeomorphic to $f^{-1}(a)/S^1$.

We will refer to $M_{[a, \infty)}$ as **the cut of M with respect to the ray $[a, \infty)$** (with the action and the function f understood).

Proof. The circle $S^1 \subset \mathbb{C}$ acts on the product manifold $M \times \mathbb{C}$ by $\lambda \cdot (m, z) = (\lambda \cdot m, \lambda^{-1}z)$. The function $\Psi(m, z) = f(m) - |z|^2$ is S^1 invariant.

The level set $\Psi^{-1}(a)$ is a (set theoretic) disjoint union of two manifolds

$$\Psi^{-1}(a) = \left\{ (m, z) : f(m) > a, \quad z = e^{i\theta} \sqrt{f(m) - a} \right\} \sqcup \{(m, 0) : f(m) = a\}.$$

The points of the first manifold are regular points of Ψ because they are regular points of the function $(m, z) \mapsto -|z|^2$. The points of the second manifold are regular points of Ψ by assumption on f . Moreover, by assumption on f and by construction the circle acts freely on $\Psi^{-1}(a)$. Therefore the quotient $\Psi^{-1}(a)/S^1$ is a smooth manifold.

The composition of $\sigma : \{m \in M \mid f(m) \geq a\} \rightarrow \Psi^{-1}(a)$, $\sigma(m) = (m, \sqrt{f(m) - a})$ with the orbit map $\Psi^{-1}(a) \rightarrow \Psi^{-1}(a)/S^1$ descends to a homeomorphism

$$\bar{\sigma} : M_{[a, \infty)} \rightarrow \Psi^{-1}(a)/S^1.$$

This gives the cut $M_{[a, \infty)}$ the structure of a smooth manifold. Moreover, $\bar{\sigma}$ restricted to $\{f(m) > a\}$ is an open embedding and $\Psi^{-1}(a)/S^1 \setminus \bar{\sigma}(\{f(m) > a\})$ is $f^{-1}(a)/S^1$. \square

Completely analogously one defines

$$M_{(-\infty, a]} := \{m \in M \mid f(m) \in (-\infty, a]\} / \sim,$$

the cut of M with respect to the ray $(-\infty, a]$. It is also a manifold.

Remark 2.2. The action of S^1 on the manifold M descends to an action of S^1 on the cut $M_{[a, \infty)}$. The restriction $f|_{f^{-1}([a, \infty))}$ descends to an S^1 invariant function \bar{f} on the cut.

More generally if a group G acts on M preserving the function f and commuting with the action of S^1 , then the cut $M_{[a, \infty)}$ has a naturally induced action of G which again commutes with the action of S^1 and preserves \bar{f} .

It is not hard to see that one does not need a global circle action for the construction in Proposition 2.1 to work.

Proposition 2.3. *Suppose M is a manifold with boundary ∂M . Suppose further that ∂M is a principal S^1 bundle. Let $X = M / \sim$, where, for $m \neq m'$, $m \sim m'$ if and only if*

1. $m, m' \in \partial M$ and
2. $m = \lambda \cdot m'$ for some $\lambda \in S^1$,

Then X is a smooth manifold, $\partial M / S^1$ is a submanifold of X , and $X \setminus (\partial M / S^1)$ is diffeomorphic to $M \setminus \partial M$.

Proof. We may assume that a neighborhood of the boundary ∂M in M is diffeomorphic to $\partial M \times [0, 1)$. Then $M = (M \setminus \partial M) \cup_{\partial M \times (0, 1)} (\partial M \times [0, 1))$.

The circle action on ∂M extends trivially to a circle action on $\partial M \times (-1, 1)$ making the projection map $f : \partial M \times (-1, 1) \rightarrow (-1, 1)$ S^1 -invariant. By Proposition 2.1, $(\partial M \times (-1, 1))_{[0, \infty)}$ is a smooth manifold. Moreover, it is easy to see that

$$X = (M \setminus \partial M) \cup_{\partial M \times (0, 1)} (\partial M \times (-1, 1))_{[0, \infty)},$$

hence is smooth as well. \square

Example 2.4. Let $M = \mathbb{T}^2 \times [0, 1]$ where \mathbb{T}^2 is the standard torus $\mathbb{R}^2/2\pi\mathbb{Z}^2$. Then $\partial M = \mathbb{T}^2 \times \{0, 1\}$. Let S^1 act on ∂M as follows: S^1 acts on $\mathbb{T}^2 \times \{0\}$ by $\lambda \cdot (x, y) = (x + \lambda \bmod 2\pi\mathbb{Z}, y)$ and on $\mathbb{T}^2 \times \{1\}$ by $\lambda \cdot (x, y) = (x, y + \lambda \bmod 2\pi\mathbb{Z})$. Then X is S^3 .

2.2. Symplectic cuts. Next suppose that additionally the manifold M possesses an S^1 invariant symplectic form ω such that the function f is a moment map for the action of S^1 . Then the cut $M_{[a, \infty)}$ is a symplectic manifold. More precisely:

Theorem 2.5. *Let (M, ω) be a symplectic manifold with a Hamiltonian action of a circle S^1 , let $f : M \rightarrow \mathbb{R}$ denote a corresponding moment map. Suppose that S^1 acts freely on the level set $f^{-1}(a)$ for some $a \in \mathbb{R}$ (so that a is a regular value of f). Then the cut of M with respect to the ray $[a, \infty)$ is naturally a symplectic manifold. Moreover, the natural embedding of the reduced space $M_a := f^{-1}(a)/S^1$ into $M_{[a, \infty)}$ is symplectic, and the complement $M_{[a, \infty)} \setminus M_a$ is symplectomorphic to the open subset $\{m \in M \mid f(m) > a\}$ of (M, ω) .*

Proof. (cf. [L]) Consider the symplectic product $(M \times \mathbb{C}, \omega + \sqrt{-1}dz \wedge d\bar{z})$. The map $\Psi(m, z) = f(m) - |z|^2$ is a moment map for an action of S^1 on $M \times \mathbb{C}$.

Arguing as in Proposition 2.1 we see that a is a regular value of Ψ and that the reduced space $\Psi^{-1}(a)/S^1$ is the cut $M_{[a, \infty)}$.

The pullback of the symplectic form $\omega + \sqrt{-1}dz \wedge d\bar{z}$ by the embedding

$$\sigma : \{m \in M : f(m) > a\} \hookrightarrow \Psi^{-1}(a), \quad \sigma(m) = (m, \sqrt{f(m) - a})$$

is ω . Consequently the induced embedding $\bar{\sigma} : \{m \in M : f(m) > a\} \hookrightarrow \Psi^{-1}(a)/S^1 = M_{[a, \infty)}$ is symplectic. Similarly one checks that the natural embedding $M_a \hookrightarrow M_{[a, \infty)}$ is symplectic as well. \square

Remark 2.6. More generally the construction can be carried out for Hamiltonian torus actions. A ray would then be replaced by a simple rational polyhedral cone in the dual of the Lie algebra of the torus with the “Delzant condition” on the edges; see [L, LMTW].

An analog of Proposition 2.3 holds as well:

Proposition 2.7. *Suppose (M, ω) is a symplectic manifold with boundary $P = \partial M$. Suppose further that P is a principal S^1 bundle, that $\omega|_P$ is S^1 invariant and that the kernel of $\omega|_P$ is precisely the vertical bundle of $P \rightarrow P/S^1$. Let $X = M/\sim$, where, for $m \neq m'$, $m \sim m'$ if and only if*

1. $m, m' \in P$ and
2. $m = \lambda \cdot m'$ for some $\lambda \in S^1$,

Then X is a symplectic manifold, P/S^1 is a symplectic submanifold of X , and $X \setminus (P/S^1)$ is symplectomorphic to $M \setminus P$.

Proof. By Proposition 2.3 X is a smooth manifold, P/S^1 is a submanifold and $X \setminus (P/S^1)$ is diffeomorphic to $M \setminus P$.

We now assume for simplicity that P is connected. Otherwise we can argue connected component by connected component.

By the equivariant coisotropic embedding theorem the product $P \times \mathbb{R}$ carries and S^1 -invariant closed 2-form $\tilde{\omega}$ (S^1 acts on $P \times \mathbb{R}$ by $\lambda \cdot (p, t) = (\lambda \cdot p, t)$) such that

1. $\tilde{\omega}|_{P \times \{0\}} = \omega|_P$
2. the S^1 action on $(P \times \mathbb{R}, \tilde{\omega})$ is Hamiltonian with a moment map $f(p, t) = t$.

Moreover there is an open S^1 equivariant embedding ψ of a neighborhood U of P in M into $P \times \mathbb{R}$ such that

1. $\psi(p) = (p, 0)$ for all $p \in P$,
2. $f \circ \psi(m) \geq 0$ for all $m \in U$ and
3. $\psi^* \tilde{\omega} = \omega$.

In particular $\tilde{\omega}$ is non-degenerate near $P \times \{0\}$.

We have $M = (M \setminus P) \cup_{U \cap (M \setminus P)} U$. By Theorem 2.5 $U_{[0, \infty)}$ is a symplectic manifold, the embedding of P/S^1 into $U_{[0, \infty)}$ is symplectic and the difference $U_{[0, \infty)} \setminus (P/S^1)$ is symplectomorphic to $U \setminus P$. Therefore $X = (M \setminus P) \cup_{(U \setminus P)} U_{[0, \infty)}$ is a symplectic manifold with the desired properties. \square

2.3. Contact cuts. We start by digressing on group actions on contact manifolds and follow the digression by recalling the definition of the moment map for a group action preserving a contact form [Al, Ge].

Proposition 2.8. *Let M be a paracompact manifold with a contact form α , let $\xi = \ker \alpha$ be the corresponding contact structure. Suppose a Lie group G acts properly on M , that is, suppose the map $G \times M \rightarrow M \times M$, $(g, m) \mapsto (g \cdot m, m)$ is proper. Suppose further that the action of G preserves the contact structure ξ .*

Then there exists a G -invariant contact form $\bar{\alpha}$ with $\ker \bar{\alpha} = \xi$.

Proof. The argument is an adaptation of Palais's proof of the existence of invariant Riemannian metrics on manifolds with proper group actions [P].

Suppose first that the group G is compact. Then there is on G a bi-invariant measure dg normalized so that $\int_G dg = 1$. We then define $\bar{\alpha}$ to be the average of α :

$$\bar{\alpha}_x := \int_G (g^* \alpha)_x dg$$

for all $x \in M$.

Now we drop the compactness assumption. None the less, for every point $x \in M$ there exists a slice S to the action of G . That is, S is $H = \text{stab}(x)$ -invariant embedded submanifold of M such that the union $G \cdot S$ of G -orbits through the points of S is open and such that for every $s \in S$ the orbit $G \cdot s$ intersects S in a single H -orbit (see [P]). Note that the stabilizer H of x is compact because the action is proper.

Since the contact structure ξ is G -invariant its annihilator $\xi^\circ \subset T^*M$ is a G -invariant line subbundle of the cotangent bundle. Hence the restriction $\xi^\circ|_{G \cdot S}$ is completely determined by the restriction $\xi^\circ|_S$. Moreover, any G -invariant section $\sigma : G \cdot S \rightarrow \xi^\circ|_{G \cdot S}$ is completely determined by its values on S :

$$\sigma(g \cdot s) = g \cdot \sigma(s)$$

for all $g \in G$, $s \in S$.

A contact form α is a nowhere vanishing section of ξ° . Given a slice S we produce an H -invariant section $\bar{\alpha}$ of $\xi^\circ|_S$ by averaging $\alpha|_S$ over H . We then extend $\bar{\alpha}$ to the open set $G \cdot S$ by the formula $\bar{\alpha}_{g \cdot s} = g \cdot \bar{\alpha}_s$. The form $\bar{\alpha}$ is well-defined on $G \cdot S$ because $(G \cdot s) \cap S = H \cdot s$ for any $s \in S$.

Next cover M by open sets U_β of the form $G \cdot S_\beta$ where S_β are slices. In the proof of [P, Theorem 4.3.1] Palais showed that this cover may be chosen to be locally finite and that there exist a G -invariant partition of unity $\{\rho_\beta\}$ subordinate to the cover. The form

$$\bar{\alpha} := \sum \rho_\beta \overline{\alpha|_{G \cdot S_\beta}}$$

is the desired invariant contact form. Here $\overline{\alpha|_{G \cdot S_\beta}}$ is the average of $\alpha|_{G \cdot S_\beta}$. \square

Definition 2.9. Let M be a manifold with a contact one-form α . Suppose a Lie group G acts properly on M and preserves the contact distribution $\ker \alpha$. By averaging if necessary (see above) we may assume that α is G -invariant. We define the corresponding **moment map** $\Phi : M \rightarrow \mathfrak{g}^*$ (\mathfrak{g}^* is the dual of the Lie algebra \mathfrak{g} of G) by the equation

$$\langle \Phi, \eta \rangle = \alpha(\eta_M)$$

for all $\eta \in \mathfrak{g}$. Here $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ is the standard pairing, and η_M denotes the vector field on M induced by $\eta \in \mathfrak{g}$.

The following result is a slight generalization of Theorem 6 in [Ge].

Theorem 2.10. Suppose (M, α) is a contact manifold with a proper action of a Lie group G preserving the contact form α . Suppose 0 is a regular value of the corresponding moment map $\Phi : M \rightarrow \mathfrak{g}^*$. Then $\alpha|_{\Phi^{-1}(0)}$ descends to a contact form α_0 on the orbifold $M_0 := \Phi^{-1}(0)/G$.

Proof. The proof is identical to the proof of Theorem 6 in [Ge]. The main idea of the proof is that a point x lies in the zero level set of the moment map if and only if the orbit $G \cdot x$ is tangent to $\ker \alpha$; hence $\alpha|_{\Phi^{-1}(0)}$ descends to a 1-form α_0 on M_0 . \square

We will refer to the pair (M_0, α_0) as the contact quotient of (M, α) or as the reduced space.

Theorem 2.11. Let (M, α) be a contact manifold with an action of S^1 preserving α and let f denote the corresponding the moment map. Suppose that S^1 acts freely on the zero level set $f^{-1}(0)$. Then the cut $M_{[0, \infty)}$ of M is naturally a contact manifold. Moreover, the natural embedding of the reduced space $M_0 := f^{-1}(0)/S^1$ into $M_{[0, \infty)}$ is contact and the complement $M_{[0, \infty)} \setminus M_0$ is contactomorphic to the open subset $\{m \in M \mid f(m) > 0\}$ of (M, α) .

Proof. Consider the contact manifold $(M \times \mathbb{C}, \alpha + \frac{1}{2}\sqrt{-1}(zd\bar{z} - \bar{z}dz))$ with the circle action $\lambda \cdot (m, z) = (\lambda \cdot m, \lambda^{-1}z)$. The map $\Psi(m, z) = f(m) - |z|^2$ is the corresponding moment map. Arguing as in Proposition 2.1 we see that 0 is a regular value of Ψ and that the reduced space $\Psi^{-1}(0)/S^1$ is the cut $M_{[0, \infty)}$.

The pullback of the contact form $\alpha + \frac{1}{2}\sqrt{-1}(zd\bar{z} - \bar{z}dz)$ by the embedding

$$\sigma : \{m \in M : f(m) > 0\} \hookrightarrow \Psi^{-1}(0), \quad \sigma(m) = (m, \sqrt{f(m)})$$

is α . Consequently the induced embedding $\bar{\sigma} : \{m \in M : f(m) > 0\} \hookrightarrow \Psi^{-1}(0)/S^1 = M_{[0, \infty)}$ is contact. Similarly one checks that the natural embedding $M_0 \hookrightarrow M_{[0, \infty)}$ is contact as well. \square

Example 2.12. Consider the three-manifold $M = S^1 \times S^1 \times \mathbb{R}$ with coordinates θ_1, θ_2 and t . The form $\alpha = \cos t d\theta_1 + \sin t d\theta_2$ on M is contact. The two torus $\mathbb{T}^2 = S^1 \times S^1$ acts on M by

$$(a, b) \cdot (\theta_1, \theta_2, t) = (\theta_1 + a, \theta_2 + b, t)$$

for all $(a, b) \in S^1 \times S^1$, $(\theta_1, \theta_2, t) \in S^1 \times S^1 \times \mathbb{R}$ (here we think of S^1 as $\mathbb{R}/2\pi\mathbb{Z}$). The action preserves the contact form α , and the corresponding moment map $f = (f_1, f_2) : M \rightarrow \mathbb{R}^2$ is given by

$$f(\theta_1, \theta_2, t) = (\cos t, \sin t).$$

Let's cut M with respect to $[0, \infty)$ using the second component $f_2 = \sin t$ of f . Since

$$f_2^{-1}([0, \infty)) = \{(\theta_1, \theta_2, t) : t \in [2\pi n, \pi(2n+1)], n \in \mathbb{Z}\},$$

the cut is $\bigsqcup_{n \in \mathbb{Z}} \{(\theta_1, \theta_2, t) : t \in [2\pi n, \pi(2n+1)]\} / \sim$, which is the disjoint union of countably many copies of $S^1 \times S^2$. We will see later that the contact structure that we have constructed on $S^1 \times S^2$ is fillable.

Remark 2.13. As in the symplectic case, if the contact manifold carries a torus action then we can define cuts with respect to a simple rational polyhedral cone in the dual of the Lie algebra of the torus. One can prove directly that if the moment map is transverse to all faces of the cone then the resulting cut space is a contact orbifold. Alternatively, one can apply Theorem 2.11 above and contact reduction in stages.

Remark 2.14. The construction of a contact cut does not depend on a choice of a contact form. Suppose we change the contact form by multiplying it by a positive invariant function. The corresponding moment map gets multiplied by the same function. Consequently the contact form on the cut space gets multiplied by a positive function.

Alternatively, since contact reduction can be defined without any reference to moment maps [S], contact cuts may also be defined without moment maps: we define it to be the contact reduction of $M \times \mathbb{C}$.

It is important to note that we have a contact analog of Proposition 2.3.

Proposition 2.15. *Suppose (\tilde{M}, α) is a contact manifold, M is a manifold with boundary of the same dimension as \tilde{M} embedded in \tilde{M} . Suppose further that there is a neighborhood U in \tilde{M} of the boundary ∂M and a free S^1 action on U preserving α such that the corresponding moment map $f : U \rightarrow \mathbb{R}$ satisfies*

1. $f^{-1}(0) = \partial M$ and
2. $f^{-1}([0, \infty)) = U \cap M$.

Let $X = M / \sim$, where, for $m \neq m'$, $m \sim m'$ if and only if

1. $m, m' \in \partial M$ and
2. $m = \lambda \cdot m'$ for some $\lambda \in S^1$,

Then X is a contact manifold, $\partial M / S^1$ is a contact submanifold of X , and $X \setminus (\partial M / S^1)$ is contactomorphic to $M \setminus \partial M$.

Proof. By Proposition 2.3 X is a smooth manifold and $X \setminus (\partial M / S^1)$ is diffeomorphic to $M \setminus \partial M$. We would like to show that X is contact.

We have $M = (M \setminus \partial M) \cup_{U \cap (M \setminus \partial M)} U \cap M$. By Theorem 2.11 $U_{[0, \infty)}$ is a contact manifold, the embedding of $\partial M / S^1$ into $U_{[0, \infty)}$ is contact and the difference $U_{[0, \infty)} \setminus (\partial M / S^1)$ is contactomorphic to $U \cap (M \setminus \partial M)$. Therefore $X = (M \setminus \partial M) \cup_{U \cap (M \setminus \partial M)} U_{[0, \infty)}$ is a contact manifold with the desired properties. \square

Example 2.16. Let \tilde{M} denote the manifold $S^1 \times S^1 \times \mathbb{R}$ with coordinates θ_1, θ_2 and t and a contact form $\alpha = \cos \frac{\pi}{2} t d\theta_1 + \sin \frac{\pi}{2} t d\theta_2$. The manifold $M = S^1 \times S^1 \times [0, 1]$ embeds naturally into \tilde{M} . The boundary of M is $\partial M = S^1 \times S^1 \times \{0, 1\}$. Let $U_1 = S^1 \times S^1 \times (-\frac{1}{2}, \frac{1}{2})$, $U_2 = S^1 \times S^1 \times (\frac{1}{2}, \frac{3}{2})$.

Consider the S^1 action on $U = U_1 \cup U_2$ induced on U_1 by the vector field $\frac{\partial}{\partial \theta_2}$ and on U_2 by $\frac{\partial}{\partial \theta_1}$. The corresponding moment map is f is given by $f|_{U_1} = \iota(\frac{\partial}{\partial \theta_2})\alpha = \sin \frac{\pi}{2} t$ and $f|_{U_2} = \iota(\frac{\partial}{\partial \theta_1})\alpha = \cos \frac{\pi}{2} t$. The set $\{f \geq 0\}$ is $\{(\theta_1, \theta_2, t) \in \tilde{M} \mid 0 \leq t < \frac{1}{2} \text{ or } \frac{1}{2} < t \leq 1\}$. The cut manifold X is easily seen to be the 3-sphere S^3 .

We will see below (example 2.19) that the induced contact structure on X is the standard contact structure on S^3 .

2.4. Cuts of hypersurfaces of contact type. It will be useful for us to have a way of establishing that the contact structure on a contact cut is fillable. Recall that a hypersurface Σ in a symplectic manifold (M, ω) is of **contact type** if there exists on a neighborhood of Σ a vector field X such that X is transverse to Σ and such that the Lie derivative of ω with respect to X is ω : $L_X \omega = \omega$. It is well known that in this case the restriction of the contraction $\iota(X)\omega$ to Σ is a contact form.

Our first step is to note that the contact quotient of an invariant hypersurface of contact type is again a hypersurface of contact type. More precisely:

Proposition 2.17. *Suppose (M, ω) is a symplectic manifold with a Hamiltonian action of S^1 and a corresponding moment map $f : M \rightarrow \mathbb{R}$. Suppose that Σ is a hypersurface in M which is preserved by the action of S^1 . Moreover, assume that there is an S^1 invariant vector field X defined in an invariant neighborhood of Σ such that X is transverse to Σ and such that the Lie derivative of ω with respect to X is ω . Then the contact form $\alpha := (\iota(X)\omega)|_\Sigma$ is S^1 invariant, and the moment map f_Σ for the action of S^1 on (Σ, α) is, up to an additive constant, the restriction of f to Σ . We may assume that the constant is zero.*

Suppose further that S^1 acts freely on $f^{-1}(0)$. Then 0 is a regular value of f and of $f|_\Sigma$, and the contact quotient Σ_0 is a hypersurface of contact type in the symplectic quotient $M_0 := f^{-1}(0)/S^1$.

Proof. The vector field X descends to a vector field X_0 on a neighborhood of Σ_0 . It is not hard to see that the reduced symplectic form ω_0 satisfies $L_{X_0}\omega_0 = \omega_0$ and that $(\iota(X_0)\omega_0)|_{\Sigma_0}$ is the reduced contact form α_0 . \square

Corollary 2.18. *Suppose (M, ω) , $\Sigma \subset M$ and $f : M \rightarrow \mathbb{R}$ are as in the proposition above.*

The cut $\Sigma_{[0, \infty)}$ is a hypersurface of contact type in the cut $M_{[0, \infty)}$, and the cut contact form on $\Sigma_{[0, \infty)}$ is $(\iota(\tilde{X})\omega_{[0, \infty)})|_{\Sigma_{[0, \infty)}}$, where $\omega_{[0, \infty)}$ is the induced symplectic form on $M_{[0, \infty)}$ and \tilde{X} is the vector field on a neighborhood of $\Sigma_{[0, \infty)}$ induced by X .

Proof. The hypersurface $\Sigma \times \mathbb{C}$ in $M \times \mathbb{C}$ is of contact type. The cut $\Sigma_{[0, \infty)}$ is the reduction of $\Sigma \times \mathbb{C}$, and the cut $M_{[0, \infty)}$ is the reduction at zero of $M \times \mathbb{C}$. Now apply Proposition 2.17. \square

Example 2.19. Consider the cotangent bundle $M = T^*\mathbb{T}^2$ of the standard two torus with the standard symplectic form. Denote the coordinates on \mathbb{T}^2 by θ_1, θ_2 and the corresponding coordinates on $T^*\mathbb{T}^2$ by $\theta_1, \theta_2, p_1, p_2$. In these coordinates the cosphere bundle with respect to the flat metric is

$$\Sigma = \{(\theta_1, \theta_2, p_1, p_2) : p_1^2 + p_2^2 = 1\} \simeq \mathbb{T}^3.$$

The contact form on Σ is

$$\alpha = \sum p_i d\theta_i|_\Sigma.$$

Consider the action of S^1 given by

$$a \cdot (\theta_1, \theta_2, p_1, p_2) = (\theta_1, \theta_2 + a, p_1, p_2)$$

The map

$$f(\theta_1, \theta_2, p_1, p_2) = p_2$$

is a corresponding moment map. It is not hard to see that $M_{[0, \infty)} = \{(\theta_1, \theta_2, p_1, p_2) : p_2 \geq 0\} / \sim$ is symplectomorphic to $T^*S^1 \times \mathbb{C}$, and that $\Sigma_{[0, \infty)} \simeq S^1 \times S^2$. Consequently the contact structure on $S^1 \times S^2$ that we obtained by cutting is fillable, hence tight [E3] (cf. example 2.12).

Next consider the action of S^1 on $T^*\mathbb{T}^2$ given by

$$b \cdot (\theta_1, \theta_2, p_1, p_2) = (\theta_1 + b, \theta_2, p_1, p_2)$$

The map

$$h(\theta_1, \theta_2, p_1, p_2) = p_1$$

is a corresponding moment map. This action of S^1 commutes with the first action of S^1 . Consequently it descends to a Hamiltonian action on $M_{[0, \infty)} \simeq T^*S^1 \times \mathbb{C}$. The corresponding moment map $\bar{h} : T^*S^1 \times \mathbb{C} \rightarrow \mathbb{R}$ is given by $\bar{h}(\theta, p, z) = p + |z|^2$ for all $(\theta, p, z) \in T^*S^1 \times \mathbb{C}$. If we now cut again we obtain $(T^*S^1 \times \mathbb{C})_{[0, \infty)} \simeq \mathbb{C}^2$. The cut of the hypersurface is the standard S^3 in \mathbb{C}^2 . It follows that the contact structure constructed in Example 2.16 is the standard tight contact structure on the 3-sphere.

2.5. Cuts and symplectization. A contact manifold (M, α) embeds naturally as a hypersurface of contact type into its **symplectization** $(M \times \mathbb{R}, d(e^t \alpha))$, where t is the coordinate on \mathbb{R} . Moreover, if there is an action of S^1 on M preserving α , then the trivial extension of the action to $M \times \mathbb{R}$ preserves the symplectic form $d(e^t \alpha)$. The two moment maps $\Psi : M \times \mathbb{R} \rightarrow \mathbb{R}$ and $\Phi : M \rightarrow \mathbb{R}$ are related by the formula

$$\Psi(m, t) = -e^t \Phi(m)$$

for all $(m, t) \in M \times \mathbb{R}$. Using this formula it is easy to see that symplectization and reduction commute:

$$((M \times \mathbb{R})_0, (d(e^t \alpha))_0) = (M_0 \times \mathbb{R}, d(e^t \alpha_0))$$

where $(d(e^t \alpha))_0$ denotes the reduced symplectic form and α_0 denotes the reduced contact form.

Since cutting amounts to reduction, we see that cuts and symplectization commute:

Proposition 2.20. *Let (M, α) be a contact manifold with an action of S^1 preserving the contact form. The symplectization $M_{[0, \infty)} \times \mathbb{R}$ of the cut $M_{[0, \infty)}$ is symplectomorphic to the cut of the symplectization $(M \times \mathbb{R})_{[0, \infty)}$.*

2.6. Contact gluing. Recall that given a symplectic manifold (M, ω) with a function $f : M \rightarrow \mathbb{R}$ generating a Hamiltonian circle action we obtain two cut manifolds $M_{[a, \infty)}$ and $M_{(-\infty, a]}$ for every regular value a of f . Conversely, the manifold (M, ω) can be reconstructed from the cuts $M_{[a, \infty)}$ and $M_{(-\infty, a]}$ by gluing them symplectically along the codimension two submanifold M_a (see [Go] for a precise description of symplectic gluing).

Symplectic gluing has a counterpart in the contact category. If N is a contact submanifold of a contact manifold (M, α) then the normal bundle ν of N is symplectic, hence has well-defined Chern classes. Suppose now that a manifold N is embedded as a codimension two contact submanifold of two contact manifolds (M_1, α_1) and (M_2, α_2) such that the first Chern classes of the corresponding normal bundles ν_1, ν_2 are the negatives of each other: $c_1(\nu_1) = -c_1(\nu_2)$. Then one can glue M_1 and M_2 along N to obtain a new contact manifold $M_1 \#_N M_2$ with the following properties:

1. It contains a hypersurface P diffeomorphic to the sphere bundle of ν_1 (and of ν_2).
2. $M_1 \#_N M_2 \setminus P$ is contactomorphic to the disjoint union of $M_1 \setminus N$ and $M_2 \setminus N$.

This construction, with the additional assumption that $c_1(\nu_i) = 0$, $i = 1, 2$, is described in detail in [Ge]. The construction also works in full generality.

Now if (M, α) is a contact manifold with a circle action preserving the contact form α and the corresponding moment map $f : M \rightarrow \mathbb{R}$, the contact cuts $M_{[0, \infty)}$ and $M_{(-\infty, 0]}$ can be glued contactly along M_0 to reconstruct M up to a contactomorphism.

3. AN APPLICATION

Recall that according to Eliashberg's classification of co-oriented contact structures on the 3-sphere there exists exactly one overtwisted co-oriented contact structure ζ_0 on S^3 which is homotopic to the standard contact structure as a 2-plane field (cf. [Ae]). Let us refer to ζ_0 as the standard overtwisted structure on S^3 .

In this section we use contact cuts to produce countably many non-conjugate 2-tori in the group of contactomorphisms of the standard overtwisted contact structure on the 3-sphere. We then remark that for any rational number p/q the same construction produces an overtwisted contact structure ζ on a lens space $L_{p/q}$ with the same property: the group of contactomorphisms of $(L_{p/q}, \zeta)$ contains countably many non-conjugate 2-tori.

Theorem 3.1. *Consider the standard overtwisted contact structure ζ_0 on the 3-sphere S^3 . The group of contactomorphisms of (S^3, ζ_0) contains countably many non-conjugate two-tori.*

Proof. As in example 2.16 let \tilde{M} denote the manifold $S^1 \times S^1 \times \mathbb{R}$ with coordinates θ_1, θ_2 and t and a contact form $\alpha = \cos t d\theta_1 + \sin t d\theta_2$. For each non-negative integer k consider an embedding ι_k of $M = S^1 \times S^1 \times [0, 1]$ into \tilde{M} :

$$\iota_k(\theta_1, \theta_2, t) = (\theta_1, \theta_2, (2\pi k + \pi/2)t).$$

Let $\alpha_k = \iota_k^* \alpha = \cos[(2k + 1/2)\pi t] d\theta_1 + \sin[(2k + 1/2)\pi t] d\theta_2$. Consider the action of S^1 on the boundary ∂M generated by $\frac{\partial}{\partial \theta_2}$ on $S^1 \times S^1 \times \{0\}$ and by $\frac{\partial}{\partial \theta_1}$ on $S^1 \times S^1 \times \{1\}$. Then as in example 2.16, the cut manifold $X = M/\sim$ is S^3 . By Proposition 2.15 each of the contact forms α_k induces a contact from $\bar{\alpha}_k$ on X .

Note that by examples 2.16 and 2.19 the contact from α_0 defines the standard tight contact structure on S^3 . We now argue that

1. All contact forms α_k are homotopic as non-vanishing one forms, that is, for all $k, l \in \mathbb{N}$ there is a family of 1-forms $\{\alpha_t^{k,l}\}_{0 \leq t \leq 1}$ with $\alpha_0^{k,l} = \bar{\alpha}_k$, $\alpha_1^{k,l} = \bar{\alpha}_l$, and $(\alpha_t^{k,l})_x \neq 0$ for all $x \in X$. Hence the corresponding contact structures are all homotopic as 2-plane fields.
2. For $k > 0$ the contact structures $\ker \bar{\alpha}_k$ are overtwisted. Since overtwisted contact structures on 3-manifolds are classified by the homotopy type of the corresponding 2-plane fields [E1], it would then follow that all the contact structures $\bar{\alpha}_k$ are equivalent provided $k > 0$.
3. For any two distinct integers $k, l \geq 1$, there is no contactomorphism $\psi : (X, \bar{\alpha}_k) \rightarrow (X, \bar{\alpha}_l)$ which is $S^1 \times S^1$ equivariant.

Items (2) and (3) together then prove the theorem.

To construct a path between $\bar{\alpha}_k$ and $\bar{\alpha}_l$ consisting of nowhere vanishing 1-forms, consider first the straight line homotopy between α_k and α_l on M . It fixes the end points and so descends to a homotopy between $\bar{\alpha}_k$ and $\bar{\alpha}_l$ on X . If the homotopy vanishes at a point x_0 at a time t_0 correct it by pushing the path out in the dt direction.

To see that the contact structures defined by $\bar{\alpha}_k$ ($k > 0$) are overtwisted, fix $c \in S^1$ and consider the subset $D_c^k \subset X$ given by

$$D_c^k = \{[\theta_1, \theta_2, t] \in X \mid \theta_1 = c, \quad 0 \leq t \leq \frac{2\pi}{4k+1}\},$$

where $[\theta_1, \theta_2, t]$ is the class of $(\theta_1, \theta_2, t) \in M$. The set D_c^k is an overtwisted disk in $(X, \bar{\alpha}_k)$.

To prove (3) consider two contact connected manifolds $(N_1, \alpha_1), (N_2, \alpha_2)$ with an action of a Lie group G preserving the contact forms. Let $f_i : N_i \rightarrow \mathfrak{g}^*$ ($i = 1, 2$) denote the corresponding moment maps. If $\psi : (N_1, \alpha_1) \rightarrow (N_2, \alpha_2)$ is a G -equivariant contactomorphism, then $\psi^* \alpha_2 = h \alpha_1$ for some nowhere vanishing G -invariant function h on N_1 . Since N_1 is connected, h is either always positive or always negative. Say $h > 0$. Then for any nonzero vector $\xi \in \mathfrak{g}^*$ the preimages $f_i^{-1}(\{e^s \xi \mid s \in \mathbb{R}\})$ of the ray through ξ have the same number of connected components (if $h < 0$ then $\pi_0(f_1^{-1}(\{e^s \xi \mid s \in \mathbb{R}\})) = \pi_0(f_2^{-1}(\{-e^s \xi \mid s \in \mathbb{R}\}))$).

Now for any $k > 0$ the map moment map $f_k : (X_k, \bar{\alpha}_k) \rightarrow \mathbb{R}^2$ for the action of $S^1 \times S^1$ is given by

$$f_k([\theta_1, \theta_2, t]) = (\cos(\frac{4k+1}{2}\pi t), \sin(\frac{4k+1}{2}\pi t))$$

Hence

$$\pi_0(f_k^{-1}(\{e^s(-1, 1) \mid s \in \mathbb{R}\})) = \pi_0(f_k^{-1}(\{e^s(1, -1) \mid s \in \mathbb{R}\})) = k.$$

Therefore (3) follows and so does the theorem. \square

Let us now turn our attention to lens spaces. Recall that the lens space $L_{p/q}$ is the manifold obtained by attaching two solid tori $S^1 \times D^2$ together by a diffeomorphism sending a meridian $\{x\} \times \partial D^2$ to a circle of slope $p/q \in \mathbb{Q}$ where we use the convention that $\{x\} \times \partial D^2$ has slope ∞ and $S^1 \times \{y\}$ has slope 0. Thus $L_{1/0} = S^1 \times S^2$ and $L_{0/1} = S^3$. Recall also that the fraction p/q determines $L_{p/q}$ completely. At a loss of some precision Theorem 3.1 can be generalized as follows for arbitrary lens spaces.

Theorem 3.2. *There exists on a lens space $L_{p/q}$ an overtwisted contact structure ζ with the property that the group of contactomorphisms of $(L_{p/q}, \zeta)$ contains countably many non-conjugate 2-tori.*

Proof. The proof is essentially the same as the proof of Theorem 3.1. We start by giving a description of lens spaces which is convenient for our purposes. Consider again $M = S^1 \times S^1 \times [0, 1]$. Consider the action of S^1 on the boundary of M defined on $S^1 \times S^1 \times \{0\}$ by the vector field $\frac{\partial}{\partial \theta_2}$ and on $S^1 \times S^1 \times \{1\}$ by the vector field $l \frac{\partial}{\partial \theta_1} - k \frac{\partial}{\partial \theta_2}$. By Proposition 2.3 the cut

$$X_{k,l} := M / \sim$$

is a manifold. Note that the standard action of $S^1 \times S^1$ on M descends to an action on $X_{k,l}$. One can show that $X_{k,l} = L_{(-k)/l}$.

Next fix $(0, 0) \neq (k, l) \in \mathbb{Z} \times \mathbb{N}$ and consider for each $j \in \mathbb{N}$ the embedding

$$\iota_j : M = S^1 \times S^1 \times [0, 1] \rightarrow S^1 \times S^1 \times \mathbb{R}, \quad \iota_j(\theta_1, \theta_2, t) = (\theta_1, \theta_2, (\theta_{k,l} + 2\pi j)t)$$

where $\theta_{k,l}$ is the unique angle with $0 < \theta_{k,l} \leq \pi$ and $\tan \theta_{k,l} = l/k$. Let $\alpha_j = \iota_j^*(\cos t d\theta_1 + \sin t d\theta_2)$. By Proposition 2.15 each of the contact forms α_j induces a contact from $\bar{\alpha}_j$ on $X_{k,l}$. As in the proof of Theorem 3.1 one shows that

1. For all $j, j' \geq 0$ the contact structures $\ker \bar{\alpha}_j$ and $\ker \bar{\alpha}_{j'}$ are homotopic as 2-plane fields.
2. The contact structures $\ker \bar{\alpha}_j$ are overtwisted for all $j > 0$, hence are all equivalent.

Let us denote the contact structure defined by $\bar{\alpha}_j$, $j > 0$ by ζ . This is the contact structure in the statement of the theorem.

Finally by examining the number of connected components of the fibers the appropriate moment maps one sees that for any two distinct integers $j, j' \geq 1$, there is no contactomorphism $\psi : (X_{k,l}, \bar{\alpha}_j) \rightarrow (X_{k,l}, \bar{\alpha}_{j'})$ which is $S^1 \times S^1$ equivariant. \square

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REFERENCES

- [Ae] B. Aebischer, M. Borer, M. Kälin, Ch. Leuenberger, H. M. Reimann, *Symplectic geometry*, Birkhäuser Verlag, Basel, 1994. xii+239 pp.
- [Al] C. Albert, Le théorème de réduction de Marsden-Weinstein en géométrie cosymplectique et de contact, *J. Geom. Phys.* **6** (1989), no. 4, 627–649.
- [E1] Y. Eliashberg, Classification of overtwisted contact structures on 3-manifolds, *Invent. Math.* **98** (1989), no. 3, 623–637.
- [E2] Y. Eliashberg, Filling by holomorphic discs and its applications, in *Geometry of low-dimensional manifolds, 2* (Durham, 1989), 45–67, London Math. Soc. Lecture Note Ser., **151**, Cambridge Univ. Press, Cambridge, 1990.
- [E3] Y. Eliashberg, Contact 3-manifolds twenty years since J. Martinet’s work, *Ann. de l’Institut Fourier* **92** (1992), 165–192.
- [E4] Y. Eliashberg, Symplectic topology in the nineties. *Differential Geom. Appl.* **9** (1998), no. 1-2, 59–88.
- [Ge] H. Geiges, Constructions of contact manifolds, *Math. Proc. Cambridge Philos. Soc.* **121** (1997), no. 3, 455–464.
- [Go] R. Gompf, A new construction of symplectic manifolds, *Ann. of Math. (2)* **142** (1995), no. 3, 527–595.
- [Gr] M. Gromov, Pseudoholomorphic curves in symplectic manifolds, *Invent. Math.* **82**, (1985), 307–347.
- [L] E. Lerman, Symplectic cuts, *Math. Research Lett.* **2** (1995), 247–258.
- [LMTW] E. Lerman, E. Meinrenken, S. Tolman and C. Woodward, Nonabelian convexity by symplectic cuts, *Topology* **37** (1998), no. 2, 245–259.
- [P] R. Palais, On the existence of slices for actions of non-compact Lie groups, *Ann. Math.* **73** (1961), no. 2, 295–322.
- [S] P. Ševera, Contact reductions, mechanics and duality, **math.DS/9903168**, <http://xxx.lanl.gov/abs/math.DS/9903168>

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